Discrete scalar fields and general relativity

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The physical meaning, the properties and the consequences of a discrete scalar field are discussed; limits for the validity of a continuous mathematical description of fundamental physics are a natural outcome of discrete fields with discrete interactions. The discrete scalar field is ultimately the gravitational field of general relativity, necessarily, and there is no place for any other fundamental scalar field, in this context.

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I. INTRODUCTION

Although it is considered that a scalar field has not been observed in nature as a fundamental field its use as such is very frequent in the modern literature, particularly in elementary particles, field theory and cosmology. Here we will apply to the scalar field the concepts and results developed in the reference [1], referred here as the paper I, where the concept of a discrete field was introduced and its wave equation and its Green's function discussed. The standard field and its formalism, which for a distinction, we always append the qualification continuous, are retrieved from an integration over the discrete-field parameters. Remarkable in the discrete field is that it has none of the problems that plague the continuous one so that the meaning and origin of these problems can be left exposed on the passage from the discrete to the continuous formalism. Although the motivations for the introduction of a generic discrete field in paper I have being made on pure physical grounds of causality, a deeper discussion about its physical interpretation have been left for subsequent papers on specific fields. This discussion will be retaken here with the simplest structure of a field, the scalar one. It would be a too easy posture to see the discrete scalar field as just an ancillary mathematical construct devoid of any physical meaning, and this vision could be re-enforced with the discrete field being just a point propagating with the speed of light. The idea of a point-like field may sound weird at a first sight but this represents the same symmetry of quantum field theory where fields and sources are equally treated

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as quantized fields. Here it is seen from a reversal and classical perspective. Besides, point-like objects was never a novelty in physics and one of the major motivations of the, nowadays so popular, string theory is of avoiding [6] infinities and acausalities in the fields produced by point sources, a problem that do not exist for the discrete field, according to the reference [2]. On the other hand, it could look peculiar, considering the still doubtful existence of a fundamental scalar field, that it be the one picked for discussing attribution of physical reality to a discrete field. This usual strategy of considering the simplest problems first, even if just as a preliminary academical exercise, produced however, unexpected bonus. It turns out from the discrete field analysis that the scalar field is, necessarily, the gravitational field of general relativity, whose character of a second-rank tensor is assured by the way the scalar field is attached to the definition of the metric tensor. After decoding the physical meaning of the scalar field sources one is led to the unavoidable conclusion that there is no place, in this context, for the existence of any other fundamental scalar field. This has deeper implications, discussed at the end of the paper.

This paper is structured in the following way. Section II, on the sake of a brief review of the mathematical definition of discrete fields, is a recipe on how to pass from a continuous to a discrete formalism, and vice-versa. The discrete scalar field, its wave equation, its Lagrangian and its energy tensor are discussed in Section III and the consequences of discrete interactions for the mathematical description of the physical world in Section IV. Calculus (integration and differentiation) which is based on the opposite idea of smoothness and continuity, has its full validity for describing dynamics restricted then to a very efficient approximation in the case of a high density of interaction points, as turns out to be the great majority of cases of physical interest. This seems to be an answer to the Wigner's pondering [3] about the reasons behind the unexpected effectiveness of mathematics on the physical description of the world. It is argued in Section V, after analysing the results of the Section IV, that the scalar field must necessarily describe the gravitational interaction of general relativity. Section VI brings the conclusions.

II. FROM CONTINUOUS TO DISCRETE

For a concise introduction of the discrete-field concept it is convenient to replace the Minkowski spacetime flat geometry by a conical projective one of an embedding (3+2) flat spacetime:

$$\{x \in R^4\} \Rightarrow \{x, x^5 \in R^5 | (x^5)^2 + x^2 = 0\}, \tag{1}$$

where $x \equiv (\vec{x}, t)$ and $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu = |\vec{x}|^2 - t^2$. So a change Δx^5 on the fifth coordinate, allowed by the constraint $(\Delta x^5)^2 + (\Delta x)^2 = 0$, is a Lorentz scalar that can be interpreted as a change $\Delta \tau$ on the proper-time of a physical object propagating across an interval Δx : $\quad \Delta x^5 = \Delta \tau$.

The constraint

$$(\tau - \tau_0)^2 + (x - x_0)^2 = 0 \tag{2}$$

defines a double hypercone with vertex at (x_0, τ_0), whilst

$$(\tau - \tau_0) + f_\mu (x - x_0)^\mu = 0 \tag{3}$$

defines a family of hyperplanes tangent to the double hypercone and labelled by their normal[1] f_μ, a constant four-vector. The intersection of the double hypercone with a hyperplane defines a f-generator tangent to f^μ ($f^\mu := \eta^{\mu\nu} f_\nu$). A discrete field is a field defined with support on this intersection (extended causality) in contraposition [1] to the continuous field, defined with support on a hypercone (local causality):

$$\phi_f(x, \tau) := \phi(x, \tau) \Big|_{\substack{\Delta\tau + f.\Delta x = 0 \\ \Delta\tau^2 + \Delta x^2 = 0}} := \phi \Big|_f. \tag{4}$$

The symbol $\big|_f$ is a short notation for the double constraint in the middle term of eq. (4). The constraint (4) induces the directional derivative (along the fibre f, the hypercone f-generator)

$$\nabla_\mu \phi_f(x, \tau) := (\partial_\mu - f_\mu \partial_\tau) \phi_f(x, \tau). \tag{5}$$

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[1]The eq. (3) can be written in R^5 as $f_M \Delta x^M = 0$, $M = 1, 2, 3, 4, 5$ with $f_M = (f_\mu, 1)$

An action for a discrete scalar field is

$$S_f = \int d^5x \left\{ \frac{1}{2} \eta^{\mu\nu} \nabla_\mu \phi_f(x,\tau) \nabla_\nu \phi_f(x,\tau) - \chi \phi_f(x,\tau) \rho(x,\tau) \right\}, \tag{6}$$

where $d^5x = d^4x d\tau$, χ is a coupling constant and $\rho(x,\tau)$ is the source for the scalar field. There can be no mass term in a discrete-field Lagrangian because it would imply on a hidden breaking of the Lorentz symmetry with non-propagating discrete solutions of the field equations. In other words no physical object could be described by such a Lagrangian with an explicit mass term. Nevertheless, as discussed in paper I, the action (6) still describes both, massive and massless fields. The mass of a massive discrete field is implicit on its propagation with a non-constant proper-time. Eq. (6) is a scale-free action with its element volume d^5x hiding the actual (1+1)-Lagrangian of a discrete field, massive or not, on a fibre f; a mass term would break its conformal symmetry [1].

Then the field equation and the tensor energy for a discrete field are, respectively,

$$\eta^{\mu\nu} \nabla_\mu \nabla_\nu \phi_f(x,\tau) = \chi \rho(x,\tau), \tag{7}$$

$$T_f^{\mu\nu} = \nabla^\mu \phi_f \nabla^\nu \phi_f - \frac{1}{2} \eta^{\mu\nu} \nabla^\alpha \phi_f \nabla_\alpha \phi_f. \tag{8}$$

They must be compared to the standard expressions for the continuous field:

$$(\eta^{\mu\nu} \partial_\mu \partial_\nu - m^2) \phi(x) = \chi \rho(x), \tag{9}$$

$$T^{\mu\nu}(x) = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi \tag{10}$$

which can be obtained from the action

$$S = \int d^4x \left\{ \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi_f \partial_\nu \phi_f - \frac{m^2}{2} \phi^2 - \chi \phi(x) \rho(x) \right\}, \tag{11}$$

So, the passage from a continuous to a discrete field formalism can be summarized in the following schematic recipe (the arrows indicate replacements):

$$\begin{cases} \{x\} \Rightarrow \{x, x^5\}; \\ \phi(x) \Longrightarrow \phi(x,\tau)\big|_f; \\ \partial_\mu \Rightarrow \nabla_\mu, \end{cases} \tag{12}$$

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accompanied by a dropping of the mass term from the Lagrangian. Moreover a discrete field requires discrete sources [1]. The continuous $\rho(x)$ is replaced by a discrete set of point-like sources $\rho(x,\tau)$. Any apparent continuity is reduced to a question of scale in the observation. $\rho(x,\tau)$ is, like $\phi_f(x,\tau)$, a discrete field defined on a hypercone generator too, which just for simplicity, is not being considered here. This is a symmetry between fields and sources: they are all discrete fields, and the current density of one is the source of the other.

In the opposite-direction passage, from discrete to continuous, the continuous field and its field equations are recuperated in terms of effective average fields smeared over the hypercone

$$\Phi(x,\tau) = \frac{1}{2\pi} \int d^4 f \delta(f^2) \Phi_f(x,\tau). \tag{13}$$

III. THE DISCRETE SCALAR FIELD

Comparing the actions of eqs. (6) and (11) one should observe that the first one contains explicit manifestations only of the constraint (3) through the use of the directional derivatives (5), but not of the constraint (2). This one is only dynamically introduced through the solutions of the field equation, like it happens also (local causality) in the standard formalism of continuous fields [7]. As a matter of fact all the information contained in the new action (6) can be incorporated in the old action (11), without the mass term, with the simple inclusion of the constraint (3)

$$S_P = \int d^4 x d\tau \delta(\Delta\tau + f.\Delta x)\left\{\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \chi\phi(x,\tau)\rho(x,\tau)\right\}, \tag{14}$$

as the very restriction to the hyperplane (3) by itself implies on the whole recipe (12). P in (14) stands for any generic fixed point, the local hypercone vertex: $P = (x_0, \tau_0)$, $\Delta\tau = \tau - \tau_0$ and $\Delta x = x - x_0$. Local causality, dynamically implemented through the field equations, imply that the field propagates on a hypercone (the lightcone, if a massless field) with vertex on P, which is an event on the world line of $\rho(x,\tau)$.

Whereas there is no restriction on $\rho(x)$ for a continuous field, for a discrete one, as already mentioned,

it must be [1] a discrete set of point sources. A continuously extended source would not be consistent as it would produce a continuous field. The source of a discrete scalar field is given by

$$\rho(x, t_x = t_z) = q(\tau_z)\delta^{(3)}(\vec{x} - \vec{z}(\tau_z))\delta(\tau_x - \tau_z), \tag{15}$$

where $z(\tau)$ is its world line parameterized by its proper time τ; $q(\tau)$ is the scalar charge whose physical meaning will be made clear later. The sub-indices in t and τ specify the respective events x, y and z. That t_x must be equal to t_z on the LHS of eq. (15) is a consequence of the deltas on its RHS and of the constraint (2). Initially, it is assumed that both $\dot{q} \equiv \frac{dq}{d\tau}$ and $\ddot{q} \equiv \frac{d\dot{q}}{d\tau}$ exist and that they may be non null. The field eq. (7) is solved by

$$\phi_f(x, \tau) = \int d^5y G_f(x - y, \tau_x - \tau_y)\rho(y, \tau_y) \tag{16}$$

with

$$\eta^{\mu\nu}\nabla_\mu\nabla_\nu G(x, \tau) = \delta^{(5)}(x). \tag{17}$$

The discrete Green's function associated to the Klein-Gordon operator is given [1] by

$$G_f(x, \tau) = \frac{1}{2}\theta(bf^4t)\theta(b\tau)\delta(\tau + f.x), \quad \vec{x}_{\mathrm{T}} = 0, \tag{18}$$

where $b = \pm 1$, and $\theta(x)$ is the Heaviside function, $\theta(x \geq 0) = 1$ and $\theta(x < 0) = 0$. The labels L and T are used as an indication of, respectively, longitudinal and transversal with respect to the space part of f: $\vec{f}.\vec{x}_{\mathrm{T}} = 0$ and $x_{\mathrm{L}} = \frac{\vec{f}.\vec{x}}{|\vec{f}|}$.

Remarkably $G_f(x, \tau)$ does not depend on anything outside its support, the fibre f, as stressed by the append $\vec{x}_{\mathrm{T}} = 0$. One could retroactively use this knowledge in the action (6) for rewriting it as

$$S_f = \int d^5x\delta^{(2)}(\vec{x}_{\mathrm{T}})\left\{\frac{1}{2}\eta^{\mu\nu}\nabla_\mu\phi_f\nabla_\nu\phi_f - \chi\phi_f(x, \tau)\rho(x, \tau)\right\}, \tag{19}$$

just for underlining that the fibre f induces a conformally invariant $(1+1)$ theory of massive and massless fields, embedded in a $(3+1)$ theory, as generically discussed in paper I. Actually, the factor $\delta^{(2)}(\vec{x}_{\mathrm{T}})$ is an output of the actions (6) or (14) (it is not necessary to put it in there by hand) and it can never be

incorporated as a factor in the definition (18) of $G_f(x, \tau)$, except under integration sign as in eqs. (16) and (19).

Then one could, just formally, use

$$\rho_{[f]}(x - z, \tau_x - \tau_z) = q(\tau)\delta(\tau_x - \tau_z)\delta(t_x - t_z)\delta(x_{\mathrm{L}} - z_{\mathrm{L}}), \tag{20}$$

where $\rho_{[f]}$ represents the source density ρ stripped of its explicit $\vec{x}_{\mathrm{T}}$-dependence, for reducing the action to

$$S_f = \int d\tau_x dt_x dx_{\mathrm{L}} \left\{ \frac{1}{2}\eta^{\mu\nu}\nabla_\mu\phi_f\nabla_\nu\phi_f - \chi\phi_f(x, \tau)\rho_{[f]}(x, \tau) \right\}, \tag{21}$$

by just omitting the irrelevant transversal coordinates. Eq. (6) then, after its output eq. (18), is formally equivalent to eq. (21).

The solutions from eq. (9) for a point source are well known massless spherical waves propagating (forwards and backwards in time) on a lightcone in contradistinction to the solutions (18) that are, massive or massless, point signals propagating (always forwards in time) on a straight line that happens to be a generator of the hypercone (2), the one that is tangent to f. Being massive or massless is determined by τ being constant or not, as discussed in paper I. For a massive field, its mass and its time-like four velocity are hidden behind a lightlike f and a non-constant τ; they only become explicit after the passage from discrete to continuous fields. But as it will be made clear in Section V, there is no point on considering a massive discrete scalar field because any discrete scalar field must be associated to the gravitational field of general relativity. So massive discrete scalar fields will not be considered here any further.

Using the eqs. (18) and (15) with $b = +1$, $f^4 \geq 1$ that is, for the emitted field, one has

$$\phi_f(x, \tau_x) = \chi \int d^5y\,\theta(t_x - t_y)\theta(\tau_x - \tau_y)\delta[\tau_x - \tau_y + f.(x - y)]q(\tau_z)\delta^4(y - z) =$$

$$= \chi \int d\tau_y\,\theta(t_x - t_y)\theta(\tau_x - \tau_y)\delta[\tau_x - \tau_y + f.(x - y)]q(\tau_z), \tag{22}$$

where an extra factor 2 accounts for a change of normalization with respect to eq. (18) as the annihilated field (the integration over the future lightcone) is being excluded. Then,

$$\phi_f(x_{\mathrm{L}}, \vec{x}_{\mathrm{T}} = \vec{z}_{\mathrm{T}}, t_x, \tau_x = \tau_z) = \chi\theta(t_x - t_z)\theta(\tau_x - \tau_z)q(\tau_z)\Big|_{f.(x-z)=0} \tag{23}$$

or for short, just

$$\phi_f(x, \tau) = \chi q(\tau)\theta(t)\theta(\tau)\Big|_f. \tag{24}$$

$\nabla\theta(t)$ and $\nabla\theta(\tau)$ do not contribute [1] to $\nabla\phi_f$, except at $x = z(\tau)$, as a further consequence of the field constraints. So, for $t > 0$ and (therefore) $\tau \geq 0$ one can write just

$$\phi_f(x, \tau_x) = q(\tau_z)\Big|_f \tag{25}$$

$$\nabla_\nu\phi_f = -f_\nu\dot{q}\Big|_f \tag{26}$$

With eq. (26) in eq. (8) one has

$$T_f^{\mu\nu}(x, \tau_x) = f^\mu f^\nu \dot{q}^2\Big|_f \tag{27}$$

The field four-momentum, given by $\int T^{\mu\nu}n_\nu d\sigma$ for a continuous field, is reduced, thanks to the omission of the transversal coordinates, to

$$p_f^\mu = T_f^{\mu\nu}n_\nu = f^\mu\dot{q}^2\Big|_f \tag{28}$$

where n is a space like four vector [1] such that $n.f = 1$. The conservation of the energy-momentum content of ϕ_f is assured by $f^2 = 0$,

$$\nabla_\mu T_f^{\mu\nu} = -2f_\mu f^\mu f^\nu \dot{q}\ddot{q}\Big|_f = 0. \tag{29}$$

It is justified naming ϕ_f a discrete field because although being a field it is not null at just one space point at a time; but it is not a distribution, a Dirac delta function, as it is everywhere and always finite. Its differentiability, in the sense of having space and time derivatives, is however assured by its dependence on τ, a known, supposedly continuous spacetime function. It is indeed a new concept of field, a very peculiar one, discrete and differentiable; it is just a finite point-like spacetime deformation

projected on a null direction, with a well defined and everywhere conserved energy-momentum. It is this discreteness in a field that allows the union of wave-like and particle-like properties in a same physical object, which implies [13] finiteness and no spurious degree of freedom (uniqueness of solutions).

IV. DISCRETE PHYSICS

From eq. (25) we see that the field ϕ_f is given, essentially, by the charge at its retarded time, i.e. the amount of scalar charge at z, the event of its creation. ϕ_f has a physical meaning in the sense of having a content of energy and momentum when and only when $\dot{q} \neq 0$. So, the emission or the absorption of a scalar field is, respectively, consequence or cause of a change in the amount of scalar charge on its source. This is so because emitting or absorbing a scalar field requires a change in the state of its source which is so poor of structure that has nothing else to change but itself, and this is fundamental for determining the scalar charge nature. The picture becomes clearer after recalling that one is dealing with discrete field and discrete interactions implying that the change in the state of a field source occurs at an isolated event. $q(\tau)$ is not a continuous function:

$$q(\tau) := \sum_i q_{\tau_{i+1}} \bar{\theta}(\tau_{i+1} - \tau) \bar{\theta}(\tau - \tau_i), \tag{30}$$

where

$$\bar{\theta}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 1/2, & \text{if } x = 0; \\ 0, & \text{if } x < 0, \end{cases} \tag{31}$$

and the sum is over the interaction points on the source world line, $i = 1, 2, 3 \ldots$.

$$q(\tau) = \begin{cases} q_{\tau_j}, & \text{if } \tau_j < \tau < \tau_{j+1}; \\ \frac{q_{\tau_{j-1}} + q_{\tau_j}}{2}, & \text{if } \tau = \tau_j; \\ q_{\tau_{j-1}}, & \text{if } \tau_{j-1} < \tau < \tau_j, \end{cases} \tag{32}$$

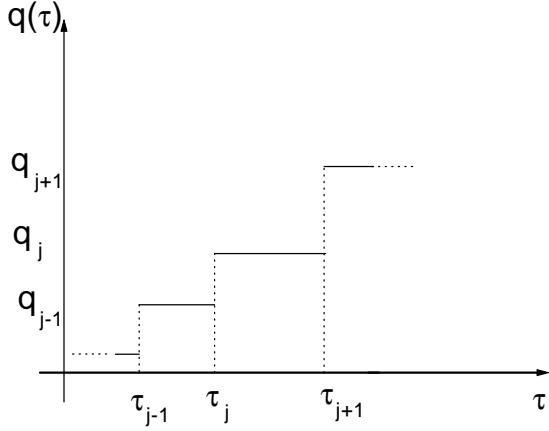as indicated in the graph of the Figure 1.

FIG. 1. Discrete changes on a discrete scalar charge along its world line. A discrete scalar charge is so poor of structure that there is nothing else to change but itself. There is change in the state of a scalar source only at the interaction points on its world line which is labelled by its proper time. If only the (discrete) interaction points are relevant the proper time may be treated as a discrete variable. In the limit of a world line densely packed of interaction points a continuous graph is a good approximation.

The change in the state of the scalar source is not null only at the interaction points and so, rigourously, it cannot be defined as a time derivative, as there is no continuous variation, just a sudden finite change. The naive use of

$$\dot{q} = q(\tau)\delta(\tau - \tau_z), \tag{33}$$

would be just an insistence on an unappropriate continuous formalism, besides artificially introducing infinities where there is none. It means that one must replace time derivatives by finite differences

$$\dot{q}(\tau) \Rightarrow \begin{cases} \Delta q_{\tau_j} & \text{if } \tau = \tau_j; \\ 0 & \text{if } \tau \neq \tau_j, \end{cases} \tag{34}$$

and a proper-time integration by a sum over the interaction points on the charge world line. The existence and meaning of any physical property that corresponds to a time derivative must be reconsidered at this fundamental level. Velocity (v) exists as a piecewise smoothly continuous function (discontinuous at the interaction points). Acceleration (a) and derivative concepts like force (F), power (P), etc rigorously do not exist. We must deal with finite differences, respectively, the sudden changes of velocity (v), momentum (p) and energy (E):

$$\begin{cases} a \Rightarrow \Delta v \\ F \Longrightarrow \Delta p \\ P \Rightarrow \Delta E \end{cases} \tag{35}$$

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So, calculus (integration and differentiation) in a discrete-interaction context becomes useless for a rigourous description of fundamental physical processes. But in practice such a strictly discrete calculus is not always feasible. What effectively counts is the scale determined by $\Delta\tau_j$, the time interval between two consecutive interaction events, face the accuracy of the measuring apparatus. The question is if $\Delta\tau_j$ is large enough to be detectable, or how accurate is the measuring apparatus used to detect it. The density of interaction points on the world line of a given point charge is proportional to the number of point charges with which it interacts. Let one consider the most favourable case of a system made of just two point charges. As the argument is supposedly valid for all fundamental interactions one can take the hydrogen atom in its ground state for consideration, treating the proton as if it were also a fundamental point particle. The order of scale of $\Delta\tau_j$ for an electron in the ground state of a hydrogen atom is given then by the Bohr radius divided by the speed of light

$$\Delta\tau_j \sim 10^{-18}s$$

which corresponds to a number of $\frac{\pi}{\alpha} \sim 400$ interactions per period (α is the fine-structure constant) or $\sim 10^{10}$ interactions/cm. So, the electron world line is so densely packed with interaction events that one can, in an effectively good descriptions for most of the cases, replace the graph of the Figure 1 by a continuously smooth curve. The validity of calculus in physics is then fully re-established as a consequence of the limitations of the measuring apparatus. The Wigner's questions [3] about the unexpected effectiveness of mathematics in the physical description of the world is recalled. The answer lies on the huge number of point sources in interaction (a sufficient condition), the large value of the speed of light and the small (in a manly scale) size of atomic and sub-atomic systems, which indirectly is a consequence of h, the Planck constant.

Although $\Delta\tau_j$ may not be measurable, at least with the present technology, the discrete formalism is justified not for replacing the continuous one where it is best, which is confirmed by high precision experiments [22] but mostly for defining and understanding its limitations. There are, besides this very generic justification, many instances of one-interaction-event phenomena, like the Compton effect, particle decay, radiation emission from bound-state systems, etc, where discrete interactions are the

natural and the more appropriate approach. These are, of course, all examples of quantum phenomena, but primarily because quantum here implies discreteness. A discrete field with discrete interactions not only requires a proper quantum context but it makes easier and natural the passage from the classical one. This is being discussed in the companion paper [4].

Discrete-continuous transition

It would be interesting to have a framework where this change from continuous to discrete interaction and vice-versa could be formally realized in a simple and direct way. One can deal with them considering the behaviour under a derivative operator of $\bar{\theta}(\tau)$ which is the mathematical description of the interaction discreteness. Then one must require that, symbolically

$$\frac{\partial}{\partial \tau} \bar{\theta}(\tau - \tau_i) := \delta_{\tau \tau_i}, \tag{36}$$

with $\delta_{\tau \tau_i}$ the Kronecker delta

$$\delta_{\tau \tau_i} = \begin{cases} 1, & \text{if } \tau = \tau_i; \\ 0, & \text{if } \tau \neq \tau_i, \end{cases} \tag{37}$$

with the meaning that at the points where the LHS of eq. (36) is not null, which are the only relevant ones, τ must be treated as a discrete variable and that the operator $\frac{\partial}{\partial \tau}$ must be seen as (or replaced by) just a sudden increment Δ and not as the limit of the quotient of two increments.

Then with such a convention one has from eq. (30) that

$$\nabla_\nu q(\tau)\Big|_f = -f_\nu \sum_i q_{\tau_i} \{\bar{\theta}(\tau_{i+1} - \tau)\delta_{\tau \tau_i} - \delta_{\tau_{i+1} \tau}\bar{\theta}(\tau - \tau_i)\} := -f_\nu \dot{q}(\tau), \tag{38}$$

which implies that $\dot{q}(\tau)$ is null when $z(\tau)$ is not a point of interaction on the charge world line. For such an interaction point τ_j one has

$$\dot{q}(\tau_j) = q_{\tau_j} \bar{\theta}(\tau_{j+1} - \tau_j) - q_{\tau_{j-1}} \bar{\theta}(\tau_j - \tau_{j-1}) = q_{\tau_j} - q_{\tau_{j-1}} \tag{39}$$

or, generically

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$$\dot{q}(\tau) = \begin{cases} \Delta q_i = q_{\tau_i} - q_{\tau_{i-1}} & \text{for } \tau = \tau_i; \\ 0 & \text{for } \tau \neq \tau_i, \end{cases} \tag{40}$$

and, from the middle term of eq. (38)

$$\nabla_\sigma^f \nabla_\nu^f q(\tau) = -2 f_\sigma f_\nu \sum_i q_{\tau_i} \delta_{\tau \tau_1} \delta_{\tau_{i+1} \tau} = 0. \tag{41}$$

In eq. (38) one has summation over the vertices and only these points on the world line contribute. That is why one has to define eq. (36). In a limit where this summation may be approximated by an integration the Kronecker delta may be replaced by a Dirac delta function and then one may have eq. (33) as a good operational approximation to eq. (40).

Then

$$\phi_f(x) = q(\tau)\Big|_f = \begin{cases} q_{\tau_{j+1}}\Big|_f & \text{if } \tau_j < \tau_{ret} < \tau_{j+1} \\ \frac{q_{\tau_{j+1}} + q_{\tau_j}}{2}\Big|_f & \text{if } \tau_{ret} = \tau_j \end{cases} \tag{42}$$

and

$$\nabla_\mu \phi_f(x) = -f_\mu \Delta q(\tau)\Big|_f = \begin{cases} -f_\mu(q_{\tau_{j+1}} - q_{\tau_j})\Big|_f & \text{if } \tau_{ret} = \tau_j \\ 0 & \text{if } \tau_{ret} \neq \tau_j \end{cases} \tag{43}$$

The field $\phi_f(x, \tau)$ is just like an instantaneous picture of its source at its retarded time. A travelling picture. If $z(\tau_{ret})$ is not a point of change in the source state, $\phi_f(x)$ does not describe a real field; its energy tensor is null. A real field always corresponds to a sudden change in its source state at its retarded time. If there is no change the field is not real, in the sense of having zero energy and zero momentum. Having no physical attribute it corresponds to the pure "gauge field" of the continuous formalism.

V. SCALAR FIELD AND GENERAL RELATIVITY

It takes an external agent on the scalar source to cause a change Δq on its charge q; a positive Δq means that a scalar field $\phi_f(x, \tau)$ has been absorbed whereas a negative one means an emission. Therefore, a discrete scalar field carries itself a charge Δq and can, consequently, interact with other

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charge carriers and be a source or a sink for other discrete scalar fields. It carries a bit of its very source, a scalar charge; it is an abelian charged field. On the other hand a new look at equations (28) and (40) reveals that $(\Delta q_j)^2$ describes the energy content of the field. So, the source of a discrete scalar field is any physical object endowed with energy which corresponds then to the scalar charge. Energy, of course, is a component of a four-vector and not a Lorentz scalar. Its four-vector character comes from the f^μ factor in eq. (28): the energy of $\phi_f(x,\tau)$ is the fourth component of the current of its squared scalar charge. The scalar charge conservation is therefore assured by and reduced to the conservation of energy and momentum given by eq. (29). Considering the relativistic mass-energy relation this implies that the discrete scalar field satisfies the Principle of Equivalence and that all physical objects interact with the scalar field through its energy-tensor. This is a form of the Principle of Universality of gravitational interaction, introduced by Moshinski [24]. So, $\phi_f(x,\tau)$ must necessarily be connected to the gravitational field. Having necessarily energy for source implies on an important consequence of uniqueness, of excluding the existence of any other distinct fundamental discrete scalar field as it must necessarily be taken as the gravitational field[2]. Moreover, as energy is not a scalar, the symmetry between discrete fields and sources, both taken as fundamental fields, implies also that there is no fundamental scalar source representing an elementary field; it must be a scalar function of a non-scalar fundamental field, like the trace of an energy tensor, for example. This lets then explicit a known symmetry of nature: the four fundamental interactions are described by gauge fields having for sources vector currents (schematically, $j = qv$), of their respective charges q, including gravity since the energy tensor is just a current of its charge, the four-vector momentum. So, this symmetry is not broken with gravity being a second-rank tensor field.

This possible physical interpretation is compatible with the General Theory of Relativity, according to the work done in references [5,25], where a discrete gravitational field defined by

$$g_{\mu\nu}^f(x) = \eta_{\mu\nu} - f_\mu f_\nu \chi \phi_f(x,\tau), \tag{44}$$

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[2]There would be no point on assuming that a same charge could be the source of two or more distinct fields, having besides the same characteristics

as a point deformation in a Minkowski spacetime, propagating on a null direction f, upon an integration on f, in the sense of eq. (26), reproduces the standard continuous solutions. That gravity be either totally [8] or partially [18,19] described by a scalar (continuous) field is an old idea [10–12], but eq. (44) implies on regarding gravity as being ultimately described by the scalar field in a metric theory. With the metric in this form the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \chi T_{\mu\nu} \tag{45}$$

is reduced [5] to

$$f_\mu f_\nu \eta^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi_f(x,\tau) = \chi T_{\mu\nu}, \tag{46}$$

as the gauge condition used in [5]

$$f^\mu\nabla_\mu\phi_f(x,\tau) = 0 \tag{47}$$

becomes an identity after eq. (26), as $f^2 = 0$.

Inherent to discrete fields, irrespective of their tensor or spinor character, is the implicit conservation of their sources as a consequence of their (discrete fields) very definition, as discussed in Section V of paper I. Whereas $T^{\mu\nu}{}_{;\mu} = 0$ is assured by the symmetry of the Einstein tensor on the LHS of eq. (45), in eq. (46) it is just a consequence (see eq.(29)) of eq. (26). This symmetry of the Einstein tensor is in this way similar to the one of the Maxwell tensor that assures charge conservation in the standard continuous-field formalism but that is a consequence of extended causality (discrete-field definition) and Lorentz symmetry [4,13] in a discrete-field approach.

The eq. (44) recalls an old derivation [23] of the field equations of general relativity by consistent re-iteration of

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \chi h(x)_{\mu\nu}, \tag{48}$$

as solution from a gauge invariant wave equation for the field $g_{\mu\nu}(x)$ in a Minkowski spacetime. The non-linearity of the Einstein's equations comes from contribution of all higher orders in $h_{\mu\nu}$ to $g_{\mu\nu}(x)$.

Therefore, the results obtained in reference [5] imply that if $h_{\mu\nu}$ is ultimately a discrete scalar field on a fibre f

$$h_{\mu\nu} = f_\mu f_\nu \phi_f(x, \tau)$$

there is no higher order contribution essentially because $f^2 = 0$. A discrete field has no self-interaction, a consequence of its definition (4) and that is explicitly exhibited in its Green's function (18). Discrete fields are solutions from linear equations. Whereas this is true for $g_{\mu\nu}^f$ of eq. (44) it is not for its f-average $g_{\mu\nu}$ of eq. (48). The non-linearity of general relativity appears here then as a consequence of the averaging process of eq. (13) that effectively smears the discrete field over the lightcone [5,25].

On the other hand the energy tensor in eq. (46) must be traceless, also a consequence of $f^2 = 0$. This recalls an old known problem in standard field theory that comes when a massless theory is taken as the $(m \to 0)-$limit of a massive-field theory [14–18], but for a discrete field, in contradistinction, a traceless tensor does not necessarily mean a massless source [1]. The wave equation (46) must be preceded by some careful qualifications, however. A discrete field requires a discrete source. The source in eq. (46) must be treated as a discrete set of point sources $T_{\mu\nu}^f(x, \tau)$ for which $f^\mu T_{\mu\nu}^f(x, \tau) = 0$. This implies that there is no exterior solution for a discrete gravitational field, only vacuum solutions. Any interior continuous solution must be seen then as an approximation for a densely packed set of point sources. From the discrete vacuum solution of eq. (46) one can, in principle, with an integration over its f-parameters, obtain any continuous vacuum solution of an imposed chosen symmetry[3] [5]. This justifies, up to a certain point, not regarding the RHS of eq. (44) as just the first two terms of a series of possible contributions from higher rank tensors. Even for a massive point-source, however, being itself a discrete field, $T_{\mu\nu}^f$ cannot be expressed in terms of its mass and of its actual four-velocity v. A traceless $T_{\mu\nu}^f(x, \tau)$ with $f^\mu T_{\mu\nu}^f(x, \tau) = 0$ does not necessarily represent a massless source nor f represents its four-velocity, as discussed in Section V of paper I.

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[3]From the superposition of the discrete fields of a spherical distribution of massless dust one retrieves the Vaydia metric [21].

The geometrical description of gravity as the curvature of a pseudo-Riemannian spacetime has its validity always assured as an absolutely good approximation due to the high density of interaction points in any real measurement, as discussed in the previous section.

VI. CONCLUSIONS

The thesis that fundamental interactions are discrete has being developed. If this is the case there is no really compelling reason for excluding gravity from such a unifying idea. It is necessary to emphasize, however, that no one is proposing the replacement of general relativity by a discrete scalar field theory of gravity. The point is that for gravity it is extremely doubtful that it could make an experimentally detectable difference. It is not a novelty that, considering the small strength of the gravitational coupling this interaction is irrelevant for physical systems involving relatively few fundamental elements. Observe that even a gravitational Aharanov-Bohm-like experiment [20] would require the gravitational field of a macroscopically large object, like the Earth. The sufficient condition for a high density of interaction points is assured and justifies continuous descriptions of gravity, of which general relativity is undoubtedly the best proposal [19]. Moreover the undectability of discrete gravity is tantamount to the unobservability of the Minkowski spacetime. At this level of approximation the Minkowski spacetime becomes the local tangent space of a curved space-time and f is a generator of the local hypercone in the tangent space. This would lead to full general relativity in accordance to a general uniqueness result [9,26] that any metric theory with field equations linear in second derivatives of the metric, without higher-order derivatives in the field equations, satisfying the Newtonian limit for weak fields and without any prior geometry must be exactly Einstein gravity itself. This reminds again the already mentioned [23] derivation of general relativity from flat spacetime but now with the distinctive aspect that the effective Riemann space comes not from a consistency requirement but as an approximation validated by a recognized limitation on the present experimental capacity. On the other hand, the knowledge of a supposedly true discrete character of all fundamental interactions, gravity included, is a permanent reminder of the limits of such a continuous approximative description, irrespective of how good their

results fit with experiments. The idea of an essential continuity of any physical interaction allows unlimited speculations that will always go beyond any level of possible experimental verifications which brings then the risk of not being able of distinguishing the reign of possibly experimentally-grounded scientific research from plain philosophical speculation or even just fiction. A discrete gravitational interaction, even if not experimentally detectable, still for a long time to come, may just make sense of existing theories as it has historically happened with all new discreteness introduced in the past, like the ideas of molecules, atomic transitions, and quarks, for example.

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